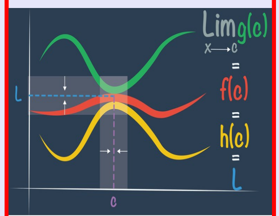
Calculus I

Lecture 46



Feb 19-8:47 AM

$f(x) = \dfrac{x^2}{\sqrt{x+1}} \geq 0$ $\qquad f'(x) = \dfrac{x(3x+4)}{2(x+1)\sqrt{x+1}}$

Domain $\quad x+1 > 0$ $\qquad f''(x) = \dfrac{3x^2+8x+8}{4(x+1)^2\sqrt{x+1}}$

$x > -1$

$(-1, \infty)$ $\qquad f'(x)=0 \rightarrow x(3x+4)=0$

$\qquad x=0 \qquad \cancel{x=-\frac{4}{3}}$

x-Int & y-Int $(0,0)$

$\lim\limits_{x \to \infty} f(x) = \infty$ $\qquad$ NO H.A. $\qquad f'(x)$ is undefined at $x=-1$

V.A. $x=-1$

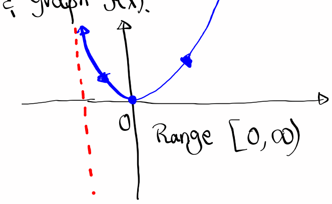
$f''(x)=0 \rightarrow 3x^2+8x+8=0$

$x = \dfrac{-8 \pm \sqrt{8^2-4(3)(8)}}{2(3)} \rightarrow$ No real Soln.

$f''(x)$ is undefined at $x=-1$

x	-1	$-.5$	0	1	∞
$f'(x)$		$-$		$+$	
$f''(x)$		$+$		$+$	
$f(x)$		$\searrow$		$\nearrow$	

Complete the Sign Chart & graph $f(x)$.



Range $[0, \infty)$

May 7-9:15 AM

Given $f(x)=\dfrac{x^3}{x^2+1}$, $f'(x)=\dfrac{x^2(x^2+3)}{(x^2+1)^2}$

1) Show $f(x)$ is an odd function. ✓

Sym. w/t origin

$$f''(x)=\frac{2x(3-x^2)}{(x^2+1)^3}$$

2) Discuss domain & V.A.

$(-\infty,\infty)$ None

3) Find x-Int & y-Int. $(0,0)$

4) Find all x-values where $f'(x)$ & $f''(x)$ are 0 or undefined.

$x=0$

$0, \pm\sqrt{3}$

May 7-9:23 AM

Use long division to $\dfrac{x^3}{x^2+1}$

$$\begin{array}{r} x \\ x^2+1 \,\big)\, x^3 + 0x^2 + 0x + 0 \\ -(x^3 + x) \\ \hline -x \end{array}$$

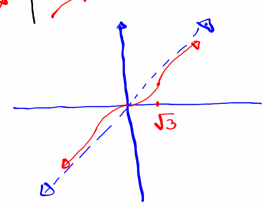
$x^2\boxed{x}=x^3$

So $\dfrac{x^3}{x^2+1}=x-\dfrac{x}{x^2+1}$ $\quad f(x)=x-\dfrac{x}{x^2+1}$

5) Find $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty}\left[x-\dfrac{x}{x^2+1}\right]=\boxed{\pm\infty}$; $\dfrac{x}{x^2+1}\to 0$

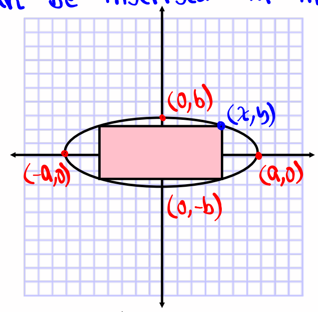
$y=x$ Slant Asymptote

x	$-\infty$	$-\sqrt{3}$		0		$\sqrt{3}$		∞
$f'(x)$	+		+	•	+		+	
$f''(x)$	+	•	−	•	+	•	−	
$f(x)$	↗		↗		↗		↗	



May 7-9:27 AM

Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Dimensions $2x$ by $2y$

Area $2x \cdot 2y = 4xy$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$LCD = a^2b^2$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$y^2 = \frac{a^2b^2 - b^2x^2}{a^2}$$

$$y = \sqrt{\frac{b^2(a^2-x^2)}{a^2}}$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$$\text{Area} = 4xy$$

$$= 4x \cdot \frac{b}{a}\sqrt{a^2-x^2}$$

$$f(x) = \frac{4b}{a} \cdot x\sqrt{a^2-x^2}$$

Maximize $f(x)$

1) $f'(x)$

2) $f'(x)=0$

3) use First derivative test to find x when $f(x)$ has max. value.

May 7-9:33 AM

$$f(x) = \frac{4b}{a} \cdot x\sqrt{a^2-x^2}$$

$$f(x) = \frac{4b}{a} \cdot \sqrt{x^2(a^2-x^2)} \rightarrow f(x) = \frac{4b}{a}\left(a^2x^2-x^4\right)^{1/2}$$

$$f'(x) = \frac{\cancel{4}^{2}b}{a} \cdot \frac{1}{\cancel{2}}\left(a^2x^2-x^4\right)^{\frac{1}{2}-1} \cdot \left(2a^2x-4x^3\right)$$

$$f'(x) = \frac{2b}{a} \cdot \frac{2x(a^2-2x^2)}{\sqrt{a^2x^2-x^4}} \qquad f'(x) = \frac{4bx(a^2-2x^2)}{a\sqrt{x^2(a^2-x^2)}}$$

$$f'(x) = \frac{4b\cancel{x}}{a\cancel{x}} \cdot \frac{a^2-2x^2}{\sqrt{a^2-x^2}} \qquad f'(x) = \frac{4b}{a} \cdot \frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

$f'(x)=0$

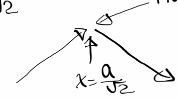
$a^2-2x^2=0$

$x^2 = \frac{a^2}{2} \qquad x = \frac{a}{\sqrt{2}}$

if $x=0 \rightarrow f'(x) > 0$

if $x > \frac{a}{\sqrt{2}} \rightarrow f'(x) < 0$

Max. pt.



max. Area when $x = \frac{a}{\sqrt{2}}$

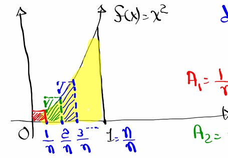
Find y → Largest Area

$4xy$

$4 \cdot \frac{a}{\sqrt{2}} \cdot \square$

May 8-9:07 AM

Find the area below $f(x)=x^2$, above x-axis from $x=0$ to $x=1$.



Take $[0,1]$ and divide it into n subintervals.

$A_1=\frac{1}{n}\cdot\left(\frac{1}{n}\right)^2$

$A_2=\frac{1}{n}\cdot\left(\frac{2}{n}\right)^2$

$A_3=\frac{1}{n}\cdot\left(\frac{3}{n}\right)^2$

$A_4=\frac{1}{n}\cdot\left(\frac{4}{n}\right)^2$

$\vdots \quad \vdots$

$A_n=\frac{1}{n}\left(\frac{n}{n}\right)^2$

Total Area

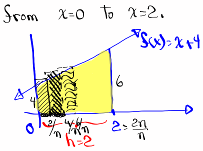
$$A = A_1+A_2+A_3+\cdots+A_n$$
$$=\frac{1}{n}\left(\frac{1}{n}\right)^2+\frac{1}{n}\cdot\left(\frac{2}{n}\right)^2+\frac{1}{n}\left(\frac{3}{n}\right)^2+\cdots+\frac{1}{n}\left(\frac{n}{n}\right)^2$$
$$=\frac{1}{n}\cdot\left[\frac{1^2}{n^2}+\frac{2^2}{n^2}+\frac{3^2}{n^2}+\cdots+\frac{n^2}{n^2}\right]$$
$$=\frac{1}{n}\cdot\frac{1}{n^2}\left[\underbrace{1^2+2^2+3^2+\cdots+n^2}\right]$$
$$=\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{2n^3+\cdots}{6n^3}$$

As $n\to\infty$, $\lim\limits_{n\to\infty}\frac{2n^3+\cdots}{6n^3}=\frac{2}{6}=\frac{1}{3}$

$$A=\lim_{n\to\infty}\frac{2n^3+\cdots}{6n^3}=\frac{1}{3}$$

May 8-9:17 AM

Find the area below $f(x)=x+4$, above x-axis from $x=0$ to $x=2$.



Trapezoid

$A=\frac{h(B+b)}{2}$

$=\frac{2(6+4)}{2}=10$ ✓

$[0,2]\to 2$ units

divide by n

$\frac{2}{n}$

Divide $[0,2]$ into n subintervals

$A_1=\frac{2}{n}\left[\frac{2}{n}+4\right]\quad A_2=\frac{2}{n}\left[\frac{4}{n}+4\right]\quad A_3=\frac{2}{n}\left[\frac{6}{n}+4\right]\cdots$

$A_n=\frac{2}{n}\left[\frac{2n}{n}+4\right]$

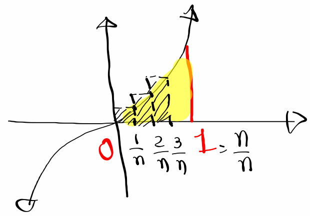
Total Area

$$A=\lim_{n\to\infty}\left[A_1+A_2+A_3+\cdots+A_n\right]$$
$$=\lim_{n\to\infty}\left[\frac{2}{n}\left(\frac{2}{n}+4+\frac{4}{n}+4+\frac{6}{n}+4+\cdots+\frac{2n}{n}+4\right)\right]$$
$$=\lim_{n\to\infty}\frac{2}{n}\left[\frac{1}{n}\overbrace{(2+4+6+\cdots+2n)}^{2}+n\cdot 4\right]$$
$$=\lim_{n\to\infty}\frac{2}{n}\left[\frac{2}{n}(1+2+3+\cdots+n)+4n\right]$$
$$=\lim_{n\to\infty}\left[\frac{4}{n^2}\underbrace{(1+2+3+\cdots+n)}+\frac{2}{\cancel{n}}\cdot 4\cancel{n}\right]$$
$$=\lim_{n\to\infty}\left[\frac{4}{n^2}\cdot\frac{n(n+1)}{2}+8\right]$$
$$=\lim_{n\to\infty}\left(\frac{4}{\cancel{n^2}}\cdot\frac{\cancel{n^2}+n}{2}+8\right)$$
$$=\frac{4}{2}+8=2+8=\boxed{10}$$

May 8-9:28 AM

Find the area below $f(x)=x^3$, above x-axis, on $[0,1]$

Ans. $\frac{1}{4}$ ✓



$A=\lim_{n\to\infty}\left[A_1+A_2+A_3+\cdots+A_n\right]$

$=\lim_{n\to\infty}\left[\frac{1}{n}\cdot\left(\frac{1}{n}\right)^3+\frac{1}{n}\cdot\left(\frac{2}{n}\right)^3+\frac{1}{n}\cdot\left(\frac{3}{n}\right)^3+\cdots+\frac{1}{n}\cdot\left(\frac{n}{n}\right)^3\right]$

$A=\lim_{n\to\infty}\left[\frac{1}{n}\cdot\frac{1^3+2^3+3^3+\cdots+n^3}{n^3}\right]$

$=\lim_{n\to\infty}\left[\frac{1}{n^4}\cdot\left(1^3+2^3+3^3+\cdots+n^3\right)\right]$

$=\lim_{n\to\infty}\left[\frac{1}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2\right]=\lim_{n\to\infty}\frac{n^4+\cdots}{4n^4}$

$=\boxed{\frac{1}{4}}$

May 8-9:43 AM

Formulas from Pre Calc or College Algebra

$\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^{n} i^3=\left[\frac{n(n+1)}{2}\right]^2$

Find $\underbrace{1^3+2^3+3^3+4^3}_{n=4}=\left[\frac{4(4+1)}{2}\right]^2=100$

$1+8+27+64=100$

Find $\underbrace{1^2+2^2+3^2+\cdots+10^2}_{n=10}=\frac{10(10+1)(2\cdot10+1)}{6}$

$=\frac{10\cdot11\cdot21}{6}$

$=385$

$1^2+2^2+3^2+\cdots+10^2=$

$1+4+9+16+25+36+49+64+81+100$

May 8-9:52 AM